Properties of the two-dimensional random-bond $\pm J$ Ising spin glass

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(Received 11 March 1998)

We develop an exact gauge-invariant method for studying the two-dimensional $\pm J$ spin glass. It is applied to the case of an arbitrary concentration of $(1-p)$ positive and p negative bonds and is thus a generalization of the more commonly studied $p=50\%$ model. The ground-state properties are examined and in particular it is shown that the spin correlation exponent η remains constant over the range $p_c<p<50\%$. The value obtained is $\eta=0.34\pm0.02$. A wide range of values for η is quoted in the literature. We indicate possible reasons for the discrepancies and indicate that there are potential advantages in doing calculations at concentrations markedly lower than 50%. [S1063-651X(98)07708-3]

PACS number(s): 05.50.+q, 64.60.Cn, 75.10.Nr

I. INTRODUCTION

It is now more than 20 years since the introduction of the Edwards-Anderson Hamiltonian [1], and this remains the basis for most theoretical work on spin glasses,

$$H=-\sum_{\langle ij\rangle} J_{ij}\sigma_i\sigma_j . \tag{1.1}$$

The J_{ij} are quenched random variables and the σ_i are Ising spins on a lattice. The status of our understanding of spin glasses is covered in a number of reviews [2–5]. For infinite range interactions [6], the model is well understood; it exhibits a rich structure of states related by an ultrametric topology [4]. The relevance of the mean-field results to models with short-range interactions is not clear, however.

One of the extensively studied models of a short-range spin glass is the $\pm J$ system [7]. The model comprises nearest-neighbor bonds of fixed magnitude but random sign, with an equal probability for the sign being positive or negative. In this paper we consider the two-dimensional generalized version of this model in which the signs can have different probabilities determined by a parameter p,

$$P(J_{ij})=p\,\delta(J_{ij}+J)+(1-p)\,\delta(J_{ij}-J). \tag{1.2}$$

For $p=0.5$, there is a phase transition at zero temperature. The properties around the transition have been studied by high-temperature expansions [8], Monte Carlo simulations [9–11], and transfer matrix methods [12,13]. A value $p=0$ in Eq. (1.2) corresponds to the pure ferromagnet. As p is increased from zero, the ferromagnetic critical temperature decreases and at a concentration p_c ferromagnetic order disappears. The best estimates put p_c at about 0.11 [12,14–17]. There is evidence [18] that the $T=0$ critical behavior exhibited by the $p=0.5$ system is maintained throughout the concentration range $p_c<p<0.5$. However, an exotic ground-state configuration (random antiphase state) at concentrations just above p_c has also been proposed [19,20].

The $T=0$ transition at $p=0.5$ is characterized by algebraically decaying correlations between spins,

$$[\langle S_0 S_R\rangle^2]_{\rm av}\sim R^{-\eta} \tag{1.3}$$

with various estimates [9,12,21–23] of the value of η. Scaling theories of ''droplet'' excitations [22,24,25] indicate that the $\pm J$ model constitutes a special universality class.

Two-dimensional Ising systems have the special property of allowing exact solutions—at least for large but finite systems in the absence of a magnetic field. There have been a number of calculations of this type [26–29] and these are generally based on the combinatorial or Pfaffian method [30,31]. Although the combinatorial method is applicable to both the Gaussian and the $\pm J$ models, it takes a particularly simple form for the latter. Saul and Kardar [26,27] develop an algorithm using integer arithmetic, and study defects, low-lying excitations, and the zeros of the partition function in the complex plane.

An alternative approach based on the combinatorial method was taken by Blackman and Poulter [28]. The focus was on the Pfaffian matrix that allows the combinatorial method to be expressed in closed form. It was shown that, particularly in the zero temperature limit, its eigenstates exhibit behavior that is ideally suited to characterizing the physics of frustrated systems. In addition, the algorithm enables certain quantities such as the ground-state free energy and entropy to be calculated exactly for very large lattices.

These approaches are attractive methods for obtaining exact results at zero temperature, which is, of course, a difficult limit to access by Monte Carlo methods. The added motivation for the work is the structure in the theory, which appears to capture the essence of the physics of at least one class of short-range frustrated systems.

In the present paper our earlier work [28] will be developed to study the two-dimensional $\pm J$ model over the concentration range $p_c<p\leqslant 0.5$. The theoretical development on

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which the method is based will be summarized in Sec. II, and the results of calculations of the energy and entropy over a range of p will be given in Sec. III. Importance was attributed to the spatial extent of the eigenstates of the Pfaffian matrix. This feature is explored in Sec. IV, and the relation to correlations between spins will be developed in Sec. V. There have been very few previous studies of the system away from $p=50\%$. We evaluate the exponent η in Eq. (1.3) and show that it remains constant over the concentration range from p_c to 50%.

II. BACKGROUND

We summarize here the key features of the method. The reader is referred to the earlier paper [28] for fuller details. New developments in the theory are given in this section and in Sec. V.

In the Pfaffian method, the partition function for the 2D Ising model on an N site square lattice is written [28,30,31]

$$Z=2^N\left[\prod_{\langle ij\rangle}\cosh(J_{ij}/kT)\right](\det D)^{1/2}. \tag{2.1}$$

D is the Pfaffian matrix referred to in the preceding section. Writing in terms of the skew-symmetric determinant rather than the Pfaffian itself means that D is the full square array (of order $4N$). The elements of D are either 0, ± 1, or $\pm\tanh(J_{ij}/kT)$. We prefer to multiply all the elements by i so that one is dealing with a Hermitian matrix and real eigenvalues. The determinant is, of course, unchanged.

The physics is contained in D and the formalism is applicable to an arbitrary set of nearest neighbor J_{ij}. With frustrated systems there is a nice separation of the eigenstates of D into those associated with frustration and the rest. At zero temperature the decoupling is complete for the ''frustration states'' (FS). The number of FS is equal to the number of frustrated plaquettes and the FS are completely localized on the frustrated plaquettes. The FS occur in pairs $|\alpha\rangle\pm i|\beta\rangle$ with eigenvalues $\pm\varepsilon$.

This decoupling of the FS occurs for any two-dimensional frustration model (e.g., Gaussian or $\pm J$). For the $\pm J$ model there is further simplification because the eigenvalues of the FS can be written in the $T\rightarrow 0$ limit in the form

$$\varepsilon=\tfrac{1}{2}X\ \exp(-2rJ/kT), \tag{2.2}$$

where r is an integer and X is a real number.

The change in ground-state energy and entropy resulting from the frustration can be written

$$\Delta F=2J\sum_f{}^{+} r_f, \tag{2.3}$$

$$S=k\sum_f{}^{+}\ln X_f, \tag{2.4}$$

where the notation indicates a summation over the positive members of pairs of FS. Equivalent to Eq. (2.4), of course, is writing the ground-state degeneracy M as $M=\Pi_f^{+}X_f$. The derivation of Eq. (2.3) is almost trivial whereas obtaining the apparently simple result in Eq. (2.4) involves some subtlety [28]. The ground-state energy F for N spins is $-2NJ+\Delta F$.

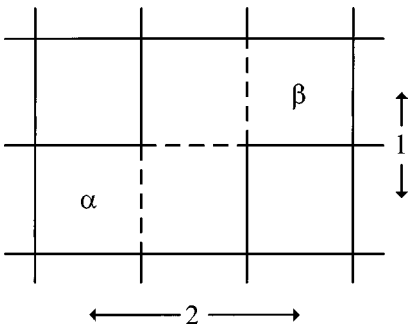

FIG. 1. Example of an elementary two frustrated plaquette configuration to illustrate definition of spatial extension. Broken lines are negative bonds and α and β label the frustrated plaquettes. Numbers indicate x and y separations.

In addition to the eigenvalues of the FS, it was postulated [28] that the form of the eigenstates themselves has physical significance. There are indications that all FS are localized for $p<p_c$ while, for $p>p_c$, a proportion of the FS is extended. The detailed behavior above p_c relates closely to physical properties such as spin-spin correlations.

One needs to take some care in the definition of the spatial extent of the FS, and this point will now be developed in a more rigorous fashion. It is instructive to consider initially a simple defect configuration in which only two frustrated plaquettes are present. An example is shown in Fig. 1. In this case the $|\alpha\rangle$ and $|\beta\rangle$ basis states are localized on the plaquettes as indicated, and ε is given by Eq. (2.2) with $X=3.0$ and $r=3$. The eigenstates corresponding to $\pm\varepsilon$ are $|\alpha\rangle\pm i|\beta\rangle$.

It is reasonable to use the Manhattan distance as a measure of the extension of this pair of frustration states, namely the sum of the x and y separation of the α and β plaquettes ($2+1=3$ in units of lattice spacings).

To formalize this, we introduce operators $\hat{x}$ and $\hat{y}$ that represent the coordinates of the plaquettes, and define expectation values

$$\langle\hat{r}\rangle=|\langle\beta|\hat{r}|\beta\rangle-\langle\alpha|\hat{r}|\alpha\rangle|, \tag{2.5}$$

where $\hat{r}$ is either $\hat{x}$ or $\hat{y}$. The Manhattan spatial extent l then is given by

$$l=\langle\hat{x}\rangle+\langle\hat{y}\rangle. \tag{2.6}$$

There is an ambiguity, however, due to an arbitrary phase factor ϕ. Eigenstates $|\alpha'\rangle\pm i|\beta'\rangle$ defined as

$$|\alpha'\rangle\pm i|\beta'\rangle=\exp(\pm i\phi)[|\alpha\rangle\pm i|\beta\rangle] \tag{2.7}$$

would be equally valid in association with $\pm\varepsilon$. Although for simple configurations like that shown in Fig. 1, the phase to use is obvious by inspection, this is no longer true for a complex configuration of frustration at arbitrary concentration p. One needs a definition of spatial extent that is not dependent on an arbitrary phase factor.

This difficulty is resolved if Eq. (2.5) is replaced by the following definition:

TABLE I. Ground-state energy F (in units of J) and entropy (in units of k) per spin for selected values of p.

p (%)	F	S
5	-1.8024 ± 0.0005	0.0103 ± 0.0002
8	-1.6924 ± 0.0005	0.0252 ± 0.0003
10	-1.6274 ± 0.0003	0.0363 ± 0.0004
11	-1.5994 ± 0.0003	0.0405 ± 0.0004
12	-1.5750 ± 0.0003	0.0440 ± 0.0004
15	-1.5182 ± 0.0003	0.0528 ± 0.0005
20	-1.4590 ± 0.0005	0.0620 ± 0.0003
50	-1.4021 ± 0.0002	0.0709 ± 0.0004

$$\langle\hat{r}\rangle=(P^2+Q^2)^{1/2}, \tag{2.8}$$

where $P=\langle\beta|\hat{r}|\beta\rangle-\langle\alpha|\hat{r}|\alpha\rangle$ and $Q=\langle\alpha|\hat{r}|\beta\rangle+\langle\beta|\hat{r}|\alpha\rangle$. The extension l is now independent of the phase factor ϕ and the overall formalism is invariant with respect to gauge transformations. Further, the definition retrieves the intuitive definition of size in the limit of simple pairs of frustrated plaquettes and no alpha-beta overlap.

III. ENERGY AND ENTROPY

Numerical calculations of the attributes of the FS can be done on large finite lattices. The procedure is one of degenerate state perturbation theory. The small quantity that appears in the perturbation expansion is $\exp(-2J/kT)$. If the perturbation theory is carried out to order $r_{\max}$ [the largest value of r that occurs among the set of FS—see Eq. (2.2)], then an *exact* solution for that configuration of disorder is obtained. Obviously $r_{\max}$ will vary between configurations but, for the sizes of lattice considered, it seldom is larger than 12. Averages are performed over many configurations of disorder. For further details about the implementation of the perturbation theory, see Ref. [28].

Square lattices of size $L\times L$ were considered. Calculations were performed with $L=64,128,256$ and the results for the energy and entropy extrapolated to an infinite lattice using the forms

$$F(L)=F+\alpha L^{-1}, \tag{3.1}$$

$$S(L)=S+\beta L^{-1}, \tag{3.2}$$

where $F(L)$ and $S(L)$ are the values for an $L\times L$ lattice and F and S are the $L\to\infty$ limits. The results obtained for a selection of values of p are shown in Table I. Between 300 and 1000 samples are used for each configurational average. Estimates for $p=50\%$ were reported previously [28]. There is a minor change in the best estimate, which is due to the improved statistics used here, but the results are in agreement within the error bars.

We observe interesting behavior in the coefficients α and β in Eqs. (3.1) and (3.2). For small values of p both α and β are negative. As p is increased both become positive. The change of sign occurs for α at around 15% while, for β, it is at about 11% (i.e., at p_c). This behavior is presumably related to the transition from localized to extended states discussed in the next section with the sign change in β exactly matching the transition.

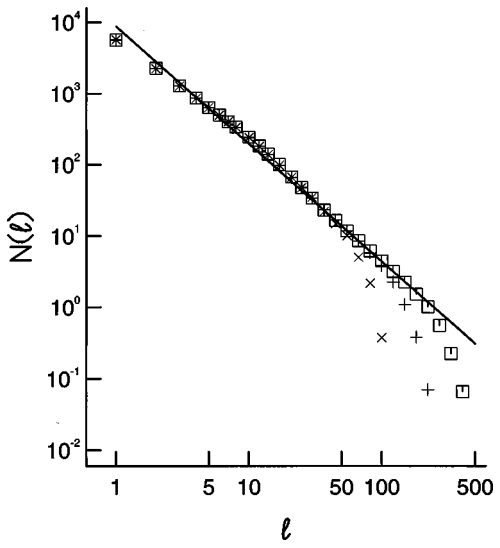

FIG. 2. Distribution $N(l)$ (normalized to a 256×256 sample size) as a function of l for $p=50\%$ for samples with $L=64$ ($\times$), 128 (+), 256 (□). A straight line fit to the $L=256$ data points is also shown.

IV. EXTENDED AND LOCALIZED STATES

It was suggested earlier [28] that p_c marks a transition between localized and extended states. Our earlier work [28] is now updated by the more rigorous definition of spatial extent given by Eq. (2.8). Some modifications are found in our earlier description, which produce a more complete picture. It is also suggested that the nature of these states in the $p>p_c$ regime relate to spin-spin correlation functions. This will be explored in the following section.

We find that, for $p>p_c$, the distribution function for the ''size'' of the FS can be described by a power law characterized by an exponent ρ

$$N(l)\sim L_1L_2l^{\rho}. \tag{4.1}$$

$N(l)$ is the number of FS whose size is larger than l for a sample of size $L_1\times L_2$. For a square lattice $(L\times L)$, the number of FS whose size is comparable to that of the lattice is $\sim L^{2-\rho}$. This approaches zero or infinity as $L\to\infty$ according to whether $\rho>\rho_c$ or $\rho<\rho_c$, where $\rho_c=2$. We thus take the condition $\rho<\rho_c$ as the criterion for the presence of extended states while, if $\rho>\rho_c$, all states are localized.

The data for $N(l)$ are plotted in Figs. 2–4 for three values of p (50%, 20%, and 12%, respectively) and, in each diagram, three values of L (64, 128, and 256). The full lines are least-squares fits to the $L=256$ data and are the best estimates of the asymptotic behavior. The slope of the lines is the same for each value of p within error bars. From this we are able to deduce a value for the exponent in Eq. (4.1) of $\rho=1.66\pm0.02$. This applies over the concentration range $11.5\%\leqslant p\leqslant50\%$. A number of other values of p in this range besides those displayed were checked to confirm the constancy.

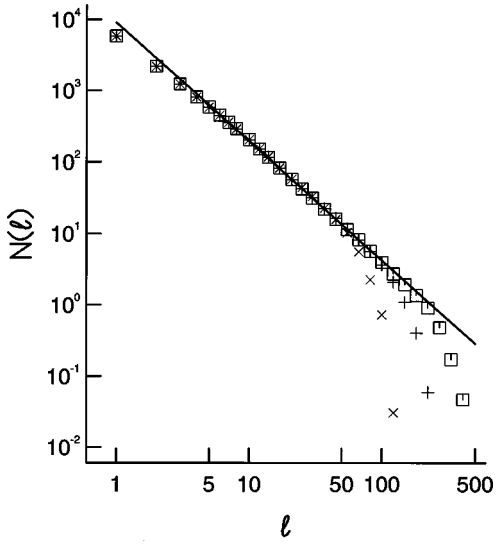

FIG. 3. Same as Fig. 2 for $p=20\%$.

Figure 5 shows the data for four values of p from 5% to 11%. It also shows the straight line fit at 50% (from Fig. 2) and at $p=11.5\%$. This latter value of p is the smallest for which $\rho=1.66$ applies. A small deviation can be observed at 11%, for which $\rho=1.73\pm0.02$, while at 10.5% a straight line fit is still possible with $\rho=1.91$ but with much larger error bars. For $p\leqslant 10.5\%$, a straight line fit is no longer feasible, but clearly there is a fast fall-off in $N(l)$ with increasing l and the states in this concentration range are certainly localized.

The conclusion then is that we have extended states characterized by an exponent ρ of 1.66 ± 0.01 for $p>p_c$, and a transition from extended to localized at p_c. The transition occurs over a concentration range of about 1% about p_c, and is presumably a finite size effect.

Preliminary work [28] at $p<p_c$ indicated that the form of Eq. (4.1) applied there as well. The current studies, which take the work to larger values of L, indicate that this is not the case. For $p<p_c$, T_c occurs at nonzero temperatures and there is no particular reason to expect special behavior at $T=0$ in this region.

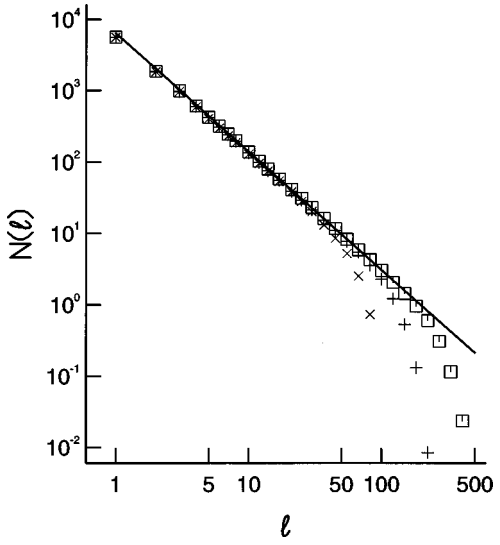

FIG. 4. Same as Fig. 2 for $p=12\%$.

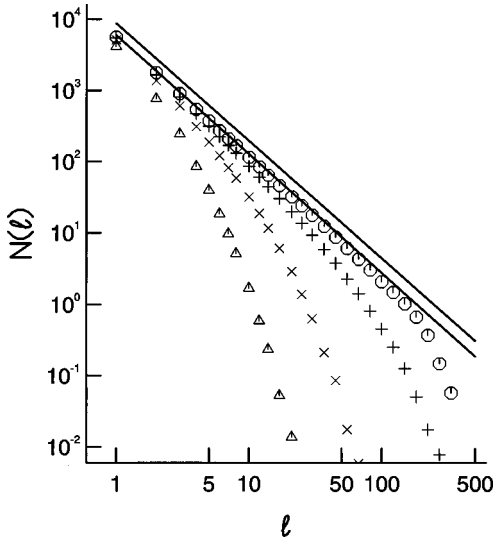

FIG. 5. Distribution $N(l)$ for $L=256$ samples as a function of l for $p=11\%$ (○), 10% (+), 8% (×), 5% (△). Straight lines obtained from fitting $L=256$ data are also shown: upper (for $p=50\%$, as in Fig. 2) and lower (for $p=11.5\%$).

V. SPIN-SPIN CORRELATIONS

Much of the information about short-range spin glasses has been obtained by examining ''droplet'' excitations [22,24,25]. One considers a closed contour in the lattice containing of the order L^2 spins; the effective block coupling J' on a length scale L is obtained from the energy cost of reversing all spins within that contour. In practice, the block coupling is usually studied by fixing the width of a strip and evaluating as a function of length the energy difference of the system with two different boundary conditions. The $\pm J$ model is special and the issue to be studied in this case is whether or not the block coupling is zero. One defines the probability $P_B(L)$ as the fraction of $L\times L$ blocks for which $J'\neq 0$. Scaling arguments suggest that $P_B(L)\sim L^{-\eta}$, where η is the exponent that characterizes the power-law decay in the zero temperature spin-spin correlation function in Eq. (1.3).

A relation between the present work and the droplet approach can be made as follows. As discussed in the preceding section, the mean number of extended frustration states in an $L\times L$ region scales as $L^{2-\rho}$. We define $P(n,L)$ as the probability that the number of extended states in the region is n. Demanding that the probability function is normalized to unity and that the mean number is $\sim L^{2-\rho}$ yields by the usual scaling arguments

$$P(n,L)=L^{-(2-\rho)}f(nL^{-(2-\rho)}), \tag{5.1}$$

where $f(x)$ is some analytic function of x.

Now the occurrence of frustration extended over a region of size $L\times L$ is associated with multiple ground-state configurations, and there will be at least one contour that can be drawn of a similar size such that the energy cost of reversing spins within that region will be zero. This can be seen in a trivial example in Fig. 1. Obviously for large L the details are much more complex and involve the cooperative effect of many frustrated plaquettes. FS that are extended determine the scale, however.

Important for the present discussion is the converse of this situation. In the absence of extended states, there will be an energy cost in such a spin reversal. It is reasonable, therefore, to expect that $P_B(L)$ and $P(n=0,L)$ scale with L in the same way. This leads us to make the identification

$$\eta = 2 - \rho. \tag{5.2}$$

It should be emphasized that we are not requiring the absence of FS within the $L \times L$ region: simply the absence of those whose size is $\sim L$. Smaller ones will influence the magnitude of a concentration (p) dependent prefactor that will appear in Eq. (1.3), but it is the extended ones that will determine η.

The conclusion of the present work then is that $\eta = 0.34 \pm 0.02$ over the concentration range $11.5\% \leqslant p \leqslant 50\%$. There is also improved accuracy over our earlier result [28] for the 50% case. The only other work that we are aware of that attempts to calculate η over this concentration range is that of Morgenstern [18]. He uses Monte Carlo methods to obtain the spin correlations directly and also obtains a constant value that, at 0.4 ± 0.1, agrees with ours within error bars. With all other work in which η is evaluated, the focus is on $p = 50\%$. Again within error bars, there is agreement with McMillan's [9] value of 0.28 ± 0.04. However, the other results reported [10,22,23,26,27] are all around 0.2.

The reason for the wide range of values reported may lie in the fact that nearly all of the calculations consider $p = 50\%$. Referring to Figs. 2–4, we see that the approach to asymptotic behavior is slower at 50%. For lower values of p (see Figs. 3 and 4), there is an excellent straight line fit to the data over the range of L of 5–100. At 50% (Fig. 2), however, one has to get above $L \sim 30$ before being in an asymptotic regime. We attempted fits to a straight line over different ranges of L and found for the lower values of p, the calculated ρ was virtually independent of the range used. This is in sharp contrast to the behavior at 50%. For an L range from ~ 30 upwards, the ρ $(=1.66)$ obtained was fully consistent with its value calculated at the lower concentrations. Including smaller values of L in the fit, however, produced values of ρ lying between 1.72 and 1.82. This would yield a corresponding η between 0.18 and 0.28 closer to values reported by a number of other authors.

VI. CONCLUSIONS

The present approach is one of a number of exact methods for treating the $\pm J$ spin glass problem. Such methods are complementary to the numerical transfer matrix and Monte Carlo techniques, and together with scaling theories have an important role to play in providing an insight into the underlying physics of these systems.

Although the method will certainly not remain exact if attempts are made to include magnetic fields or to extend it to higher dimensions, developments can be made in the exact treatment of other short-range systems. There are of course additional features within the $\pm J$ model that we have exploited fully here and in the previous work [28].

The new work in this study is the exploration of the full concentration range $p_c \leqslant p \leqslant 50\%$. A range of values for the exponent η have been reported by various workers at $p = 50\%$. In view of the comments in the preceding section concerning a possible reason for this discrepancy, it would be interesting to see what values would be obtained by alternative methods at concentrations other than 50%.

Preliminary calculations on a frustrated triangular lattice indicate a range of parameters over which a value of ρ (and hence η) occurs that is similar to that obtained here. This will be further evidence for a distinct $\pm J$ model universality class.

ACKNOWLEDGMENTS

One of us (J.R.G.) acknowledges support from the Brazilian agency CNPq. We also thank the National Electronics, the Computer Technology Centre (NECTEC), and the National Science and Technology Development Agency (NSTDA), Thailand, for use of their high performance computing facilities.

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